ON THE STABILITY OF CIRCULAR MOTIONS IN THE PROBLEM OF TWO FIXED CENTERS

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Closed periodic motions in this problem are well known. The stability of such motions has also been studied; we will point, for example, to the proof of Demin [1] of the stability of elliptical motions in the plane of both centers, in relation to the major semi-axis and the eccentricity of the ellipse.

1. General form of the stability conditions. Assume that the free material particle of a unit mass moves under the influence of forces, the potential function of which in cylindrical coordinates r, ψ, z has the form $V = \varphi(r, z)$. The kinetic energy

$$T = 1/2\ (r'^2 + r^2\psi'^2 + z'^2)$$

and the potential function do not depend directly on ψ and therefore ψ is a cyclic coordinate, to which corresponds the following integral

$$p = r^2\psi' = \beta = \text{const} \tag{1.1}$$

From this, the Routh function has the form

$$R = \frac{1}{2}(r'^2 + z'^2) - \varphi - \frac{\beta^2}{2r^2}$$

and the equations of motion of the point are

$$r'' + \varphi_r - \frac{\beta^2}{r^3} = 0, \qquad z'' + \varphi_z = 0, \qquad p' = 0 \tag{1.2}$$

where φ_r, φ_z are, respectively, the derivatives of the function φ with regard to r and z. For a stable motion, the following conditions should be satisfied: $\partial R/\partial r = 0$, $\partial R/\partial z = 0$. Suppose that these conditions are satisfied by

$$r = r_0, \qquad z = z_0 \tag{1.3}$$

for which, of course, conditions should be met

$$(\varphi_r)_0 - \frac{\beta^2}{r_0^3} = 0, \qquad (\varphi_z)_0 = 0, \qquad \psi'' = 0 \tag{1.4}$$

Here, and in the following, the index 0 in parenthesis denotes the value of the quantity at the point $r = r_0$, $z = z_0$. Using the theorem by Routh, Chetaev [2] uncovered for the stationary motion (1.3) the following sufficient condition of stability:

$$(\varphi_{rr})_0 + \frac{3}{r_0}(\varphi_r)_0 > 0, \qquad \left[(\varphi_{rr})_0 + \frac{3}{r_0}(\varphi_r)_0\right](\varphi_{zz})_0 - (\varphi_{rz})_0^2 > 0 \tag{1.5}$$

The theorem of Routh provides for a stability with regard to quantities r, r', z, z' under condition that the integral (1.1) is not changed. It is known, however, [3, 4] that the last limitation is not essential. We will show that the conditions (1.5) are also necessary ones. Let us perturb the stationary motion (1.3)

$$r = r_0 + \xi, \qquad z = z_0 + \eta, \qquad p = \beta + \zeta \tag{1.6}$$

Using the equations of motion (1.2) and (1.4), we obtain the following equations in the first approximation for the forced motion:

$$\frac{d^2\xi}{dt^2} + \left[(\varphi_{rr})_0 + \frac{3}{r_0}(\varphi_r)_0\right]\xi + (\varphi_{rz})_0\,\eta = 0, \qquad \frac{d^2\eta}{dt^2} + (\varphi_{rz})_0\,\xi + (\varphi_{zz})_0\,\eta = 0, \qquad \frac{d\zeta}{dt} = 0 \tag{1.7}$$

The characteristic equation of the system (1.7)

$$\lambda\,[\lambda^4 + (a + c)\lambda^2 + (ac - b^2)] = 0, \qquad a = (\varphi_{rr})_0 + \frac{3}{r_0}(\varphi_r)_0, \qquad b = (\varphi_{rz})_0, \qquad c = (\varphi_{zz})_0 \tag{1.8}$$

has the following roots

$$\lambda_1 = 0, \qquad \lambda_{2,3,4,5} = \pm\left(-\frac{a+c}{2} \pm \sqrt{\frac{(a+c)^2}{4} - (ac - b^2)}\right)$$

Let

$$a < 0, \qquad ac - b^2 < 0 \tag{1.9}$$

Then Equation (1.8) will have a positive root if at least one of the inequalities (1.9) is satisfied. From the theory of Liapunov on the instability with regard to the first approximate trial, the instability of a stationary motion (1.3) follows when conditions (1.9) are satisfied. Thus, the necessity of conditions (1.5) is proved.

2. The stability in the problem of two fixed centers. Let us denote the line connecting two fixed centers M_1 and M_2 subject to neutral attraction as the central line. Let us assume that M_1 has a mass m_1, and M_2 has a mass m_2. We shall select the coordinate system in the following way: the center of coordinates will be on the central line between the points M_1 and M_2 at a distance of c_1 and c_2 from them, respectively, the z-axis will be directed along the central line toward M_1 and the plane xy will be perpendicular to it. The coordinates of the centers of attraction will be $M_1(0,\ 0,\ c_1)$, $M_2(0,\ 0-c_2)$. In cylindrical coordinates

$$\varphi = -\frac{fm_1}{\sqrt{r^2+(z-c_1)^2}} - \frac{fm_2}{\sqrt{r^2+(z+c_1)^2}} \tag{2.1}$$

where f is the constant of attractive forces. If the following conditions are fulfilled

$$\psi' = \omega = \text{const}, \qquad \psi'^2 = \omega^2 = \frac{fm_1}{(r_0^2+c_1^2)^{3/2}} + \frac{fm_2}{(r_0^2+c_2^2)^{3/2}}, \qquad \frac{m_1c_1}{(r_0^2+c_1^2)^{3/2}} = \frac{m_2c_2}{(r_0^2+c_2^2)^{3/2}}$$

then, in the plane xy circular motions of the point $r = r_0$, $z = 0$, will be possible.

In addition, since the distance between the centers $M_1M_2 = c$ is constant,

$$c_1 + c_2 = c \tag{2.2}$$

Let us find the conditions of stability of the circular motion (2.2). The necessary derivatives at point $r = r_0$, $z = 0$ will have the following values:

$$(\varphi_r)_0 = \frac{fm_1r_0}{(r_0^2+c_1^2)^{3/2}} + \frac{fm_2r_0}{(r_0^2+c_0^2)^{3/2}}, \qquad (\varphi_{rr})_0 = \frac{fm_1(c_1^2-2r_0^2)}{(r_0^2+c_1^2)^{5/2}} + \frac{fm_2(c_2^2-2r_0^2)}{(r_0^2+c_2^2)^{5/2}}$$

$$(\varphi_{zz})_0 = \frac{fm_1(r_0^2-2c_1^2)}{(r_0^2+c_1^2)^{5/2}} + \frac{fm_2(r_0^2-2c_2^2)}{(r_0^2+c_2^2)^{5/2}}, \qquad (\varphi_{rz})_0 = 3r_0\left[\frac{fm_1c_1}{(r_0^2+c_1^2)^{5/2}} - \frac{fm_2c_2}{(r_0^2+c_2^2)^{5/2}}\right] \tag{2.3}$$

The first of conditions (1.5) is satisfied in the case here considered. Indeed,

$$(\varphi_{rr})_0 + \frac{3}{r_0}(\varphi_r)_0 = \frac{fm_1(r_0^2+4c_1^2)}{(r_0^2+c_1^2)^{5/2}} + \frac{fm_2(r_0^2+4c_2^2)}{(r_0^2+c_2^2)^{5/2}} > 0$$

The second condition after the division of the inequality by f^2 yields

$$\frac{m_1^2[r_0^4-7r_0^2c_1^2-8c_1^4]}{(r_0^2+c_1^2)^5} + \frac{m_2^2[s_0^4-7r_0^2c_0^2-8c_2^4]}{(r_0^2+c_2^2)^5} + $$
$$+ 2\frac{m_1m_2[r_0^4+r_0^2e_1^2+r_0^2c_2^2+9r_0^2c_1c_2-8c_1^2c_2^2]}{[(r_0^2+c_1^2)(r_0^2+c_2^2)]^{5/2}} > 0 \tag{2.4}$$

The parameters r_0, c_1 and c_2 are connected through relations (2.1) and (2.2). One can eliminate two parameters, for example, r_0 and c_2 and then (2.4) will depend only on c_1.

3. Example. The potential of the earth ellipsoid may be approximated by the potential of two fixed centers which are located on the axis of revolution of the earth, whereby one center exerts an attractive force and the other a repulsive force (a negative mass). If we adopt as a unit of distance the radius of the earth $R = 6371$ km, for the unit of mass the mass of the earth $M = 0.5974 \times 10^{27}$, and for the unit of time the solar cycle [day], and if we assume that one of the centers (repulsive) is located at a distance of $a_2 = 1$, then

$$m_1 = 1.1, \qquad m_2 = -0.1, \qquad a_1 = 0.09$$

For circular motions of a point in a plane perpendicular to the earth axis and passing at a distance $\kappa = 0.052$ from the center of the earth, $C_1 = 0.038$, $C_2 = 0.948$, $r_0 = 1.14$. The height h of the point over the surface of the earth $h \sim 900$ km. The condition (2.4) yields $0.0034 > 0$, i.e. the motion is stable.

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